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THE PROBLEM OF THE BENDING OF A BEAM LYING ON AN ELASTIC BASE†

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The plane contact problem of the transmission of a normal force of specified strength onto an elastic anisotropic, wedge-shaped plate by an elastic beam of variable flexural stiffness is considered. The beam is coupled to one of the edges of the plate and its other edge is stress-free. The solution of the problem is obtained in closed form by reducing it to a Karleman boundary-value problem with shear for a strip. A conclusion is reached concerning the nature of the discontinuity of the normal contact stress at the vertex of the wedge.

Contact problems of the interaction of elastic bodies of different shape with thin elastic elements in the form of stringers, beams or inclusions have been considered in [1–3]. Problems for an elastic isotropic or anisotropic wedge, reinforced with elastic elements of constant stiffness [4–8], and, also, the problem for an elastic isotropic wedge reinforced along the bisectrix by an elastic rod of variable stiffness [9] have been investigated using boundary-value problems in the theory of analytic functions. The contact problem for an anisotropic wedge-shaped plate with an elastic mounting of variable stiffness has been considered in [10].

We will assume that a beam with stiffness $D(x)$ lies on one boundary ($\arg z = 0$) of an elastic anisotropic body which occupies an angle $-\theta \le \arg z \le 0$ in the $z = x + iy$ plane and that a distributed normal load of strength $P_0(x)$ is applied to the beam. We shall assume that $P_0(x)$ is a bounded summable function, equal to zero outside a certain interval. There is no friction between the beam and the wedge. The other boundary of the wedge ($\arg z = -\theta$) is stress-free, $0 < \theta < 2\pi$.

The problem reduces to the following problem of the equilibrium of an elastic angle

$$\frac{d^2}{dx^2}D(x)\frac{d^2 \upsilon}{dx^2} = P_0(x) - P(x), \quad \tau_{xy}(x, 0) = 0, \quad x > 0; \quad D(x) = \frac{E_0(x)h^3(x)}{12(1-\nu_0^2)} \tag{1}$$

$$X_n(t) = Y_n(t) = 0, \quad \arg t = -\theta \tag{2}$$

where $P(x)$ is the required contact stress, which satisfies the equilibrium conditions

$$\int_0^\infty P(t)dt = \int_0^\infty P_0(t)dt = P_0, \quad \int_0^\infty tP(t)dt = \int_0^\infty tP_0(t)dt = M_0 \tag{3}$$

$E_0(x)$ is the modulus of elasticity of the beam, $h(x)$ is its thickness, ν_0 is Poisson's ratio and $\upsilon(x)$ is the vertical displacement of the points of the beam.

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We will consider the two planes of the complex variables: $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$ which are obtained from the $z = x + iy$ plane by the corresponding affine transformations

$$x_1 = x+\alpha_1 y, \quad y_1 = \beta_1 y, \quad x_2 = x+\alpha_2 y, \quad y_2 = \beta_2 y; \quad \beta_1>\beta_2>0$$

Using these transformation, the domain $S(-\theta \le \arg z \le 0)$ of the plane of the variable z transfers respectively into the domain $S_k(-\theta_k \le \arg z_k \le 0)$ of the plane of the variable z_k, $(k = 1, 2)$, $\mathrm{tg}\theta_k = \beta_k \sin\theta(\cos\theta - \alpha_k \sin\theta)^{-1}$.

If the roots of the characteristic equation $s_1 \ne s_2$, on the basis of the well-known formulae [11], the problem reduces to finding the functions $\Phi_1(z_1)$ and $\Phi_2(z_2)$, which are holomorphic in the domains S_1 and S_2 respectively with the following boundary conditions

$$\begin{aligned}&(s_1-\bar{s}_2)t_1\Phi_1(t_1)+(\bar{s}_1-\bar{s}_2)\bar{t}_1\overline{\Phi_1(t_1)}+(s_2-\bar{s}_2)t_2\Phi_2(t_2) = 0\\ &t_k = \rho(\cos\theta - s_k\sin\theta), \quad \rho = |t|>0\end{aligned} \tag{4}$$

$$(s_1-\bar{s}_2)\Phi_1(t)+(\bar{s}_1-\bar{s}_2)\overline{\Phi_1(t)}+(s_2-\bar{s}_2)\Phi_2(t) = -\bar{s}_2 P(t), \quad t>0 \tag{5}$$

$$2\mathrm{Re}[q_1\Phi_1'(x)+q_2\Phi_2'(x)] = \frac{1}{D(x)}\int_0^x dt\int_0^t [P_0(s)-P(s)]ds, \quad x>0 \tag{6}$$

It is required of the functions $\Phi_1(z_1)$ and $\Phi_2(z_2)$ that they should satisfy the conditions

$$\lim z_k\Phi_k(z_k)\to 0, \quad z_k\to 0, \quad k = 1,2$$

and, for sufficiently large $|z_k|$, have the form

$$\Phi_k(z_k) = \gamma_k/z_k + O(1/z_k), \quad k = 1,2 \tag{7}$$

$$\Phi_k(z_k) = \frac{1}{\sqrt{2\pi}z_k}\int_{-\infty}^{\infty}\frac{A_k(t)}{t}e^{it\ln z_k} - i\sqrt{\frac{\pi}{2}}\frac{A_k(0)}{z_k}, \quad z_k\in S_k \tag{8}$$

Moreover, $A_k(0)$ satisfy the condition

$$(s_2-\bar{s}_2)A_2(0) = (\bar{s}_2-s_1)A_1(0)+(\bar{s}_1-\bar{s}_2)\overline{A_1(0)}$$

Introducing the values (8) into boundary conditions (4) and (5), we obtain

$$A_k(t) = [\bar{s}_k(s_2-\bar{s}_2)e^{i(3-2k)\mu t}+\bar{s}_{3-k}(\bar{s}_k-s_{3-k})e^{-(3-2k)\delta t}+s_{3-k}(\bar{s}_2-\bar{s}_1)e^{-\gamma t}]\frac{tN(t)}{2\Delta(t)} \tag{9}$$

where

$$\mu = \ln\left|\frac{\cos\theta - s_1\sin\theta}{\cos\theta - s_2\sin\theta}\right|, \quad \delta = \theta_1-\theta_2, \quad \gamma = \theta_1+\theta_2$$

$$\Delta(t) = |s_1-s_2|^2\mathrm{ch}\gamma t - |s_1-\bar{s}_2|^2\mathrm{ch}\delta t + 4\beta_1\beta_2\cos\mu t, \quad N(t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}P(e^s)e^s e^{-its}ds$$

The first equality of (3) gives

$$\sqrt{2\pi}N(0) = \int_{-\infty}^{\infty}P(e^s)e^s ds = \int_0^{\infty}P(t)dt = P_0$$

Taking the limit in equalities (9) as $t \to 0$, we obtain

$$A_k(0) = \frac{2(-1)^k\mu\beta_2\bar{s}_k + (-1)^k\delta\bar{s}_{3-k}(\bar{s}_k - s_{3-k}) + \gamma s_{3-k}(\bar{s}_1 - \bar{s}_2)}{|s_1 - s_2|^2\gamma^2 - |s_1 - \bar{s}_1|^2\delta^2 - 4\beta_1\beta_2\mu^2}\frac{P_0}{\sqrt{2\pi}}$$

Introducing the values of the functions $\Phi_k(z_k)$, represented by formula (8), into boundary condition (6) and bearing in mind equality (9), we will have

$$2\mathrm{Re}[q_1\Phi_1'(x) + q_2\Phi_2'(x)] = \frac{1}{\sqrt{2\pi}x^2}\int_{-\infty}^{\infty}\frac{(it-1)(\Delta_2 + i\Delta_1)N(t)e^{it\ln x}}{\Delta(t)}dt + \frac{c}{x^2} \tag{10}$$

where

$$c = \sqrt{\pi}\mathrm{Im}[q_1A_1(0) + q_2A_2(0)]$$

$$\Delta_1(t) = a_1^+\mathrm{sh}\gamma t + a_1^-\mathrm{sh}\delta t + c_1^-\sin\mu t, \quad \Delta_2(t) = a_2^+\mathrm{ch}\gamma t + a_2^-\mathrm{ch}\delta t + c_1^+\cos\mu t$$

$$c_1^{\pm} = 2\beta_2\mathrm{Im}[\bar{q}_1s_1] \pm 2\beta_1\mathrm{Im}[\bar{q}_2s_2]$$

$$a_2^- + ia_1^- = (\bar{q}_1s_2 - q_2\bar{s}_1)(s_1 - s_2), \quad a_2^+ + ia_1^+ = (\overline{q_1s_2} - \overline{q_2s_1})(s_2 - s_1)$$

Substituting the values of q_1 and q_2 [11] into these formulae, carrying out some reduction and applying Vieta's theorem, we obtain

$$a_1^+ = a_{22}|s_1 - s_2|^2\mathrm{Im}\left(\frac{1}{\bar{s}_1} + \frac{1}{\bar{s}_2}\right), \quad a_1^- = a_{22}|s_1 - \bar{s}_2|^2\mathrm{Im}\left(\frac{1}{\bar{s}_2} - \frac{1}{\bar{s}_1}\right)$$

$$a_2^+ = a_2^- = c_1^+ = 0, \quad c_1^- = 4\beta_1\beta_2\mathrm{Re}\left(\frac{1}{s_1} - \frac{1}{s_2}\right), \quad \Delta_2(t) = 0$$

where a_{22} is one of the constants of elasticity of the plate.

Consequently, according to formula (10), condition (6) takes the form

$$-\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}G(t)N(t)e^{it\ln x}dt + \frac{x^2}{D(x)}\int_0^x dt\int_0^t[P(s) - P_0(s)]ds = c; \quad G(t) = \frac{(1-it)\Delta_1(t)}{\Delta(t)} \tag{11}$$

The functions $\Delta(t)$ and $\Delta_1(t)$ do not vanish anywhere except at the point $t = 0$. The point $t = 0$ is a second-order zero for the function $\Delta(t)$ and a first-order zero for the function $\Delta_1(t)$.

We put $D(x) = d_0x^{p+2}$, $d_0 > 0$ where p is any real number. After substituting $\xi_0 = \ln x$ into formula (11), differentiating both sides of the resulting equality and carrying out an inverse Fourier transformation, we obtain

$$d_0t(p + it)G(t)\Psi(t) + \Psi(t - ip) = F(t), \quad -\infty - i\varepsilon < t < \infty - i\varepsilon \tag{12}$$

where

$$\Psi(t) = \frac{N(t) - N_0(t)}{t}, \quad F(t) = -d_0G(t)(p + it)N_0(t), \quad N_0(t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}e^sP_0(e^s)e^{-its}ds$$

and ε is a positive number which may be as small as desired.

The following problem arises: to find a function, which is homomorphic in the strip $-p - \varepsilon < \mathrm{Im}z < -\varepsilon$, vanishes at infinity, is continuable in the boundary of the strip and satisfies condition (12).

The function $F(t)$ is analytically extendable in the strip $0 < \mathrm{Im}z < p$ except for the points which are the roots of the function $\Delta(t)$, where it has poles, and it vanishes at infinity.

Suppose $p > 0$. Then, the coefficients of the problem can be given the form

$$\frac{t(t-ip)(t+i)\Delta_1(t)}{\Delta(t)} = it(t^2+p^2)T_p(t)\frac{\Delta_1(t)}{\Delta(t)}\operatorname{th}\frac{\pi}{2p}t\frac{\operatorname{sh}\frac{\pi}{2p}(t-ip)}{\operatorname{sh}\frac{\pi}{2p}t}; \quad T_p(t) = \frac{t+i}{t+ip}$$

We consider the function

$$G_p(t) = T_p(t)U_p(t)$$

where

$$U_p(t) = \frac{\Delta_1(t)}{a\Delta(t)}\operatorname{th}\frac{\pi}{2p}t, \quad a = \lim_{t\to\infty}\frac{\Delta_1(t)}{\Delta(t)} = a_{22}\operatorname{Im}\left(\frac{1}{\bar{s}_1}+\frac{1}{\bar{s}_2}\right)$$

The function $G_p(t)$ is continuous along the whole of the axis and $G_p(-\infty) = G_p(+\infty) = 0$. The function $U_p(t)$ takes positive values and the function $T_p(t)$ has a single zero and one pole in the lower half plane and, therefore, $\operatorname{Ind} G_p(t) = 0$. The branch of the function $\ln G_p(t)$ which vanishes at infinity is integrable along the whole of the axis.

On the basis of results obtained earlier in [12], the functions $G_p(t)$, t^2+p^2 and the number ad_0 can be represented in the form

$$G_p(t) = \frac{X_p(t-ip)}{X_p(t)}, \quad t^2+p^2 = \frac{X_1(t-ip)}{X_1(t)}, \quad ad_0 = \frac{X_2(t-ip)}{X_2(t)}; \quad -\infty-i\varepsilon < t < (\infty-i\varepsilon) \tag{13}$$

where

$$X_p(z) = \exp\left\{\frac{1}{2i\rho}\int_{-\infty-i\varepsilon}^{\infty-i\varepsilon}\ln G_p(t)\operatorname{cth}\pi(t-z)dt\right\}$$

$$X_1(z) = p^{2iz/p}\Gamma(1+iz/p)/\Gamma(2-iz/p), \quad X_2(z) = \exp(i(z/p)\ln(ad_0)); \quad -p-\varepsilon < \operatorname{Im}z < -\varepsilon$$

Substituting expressions (13) into formula (12), we obtain

$$\begin{aligned} &\frac{\Psi(t)}{X(t)}+\frac{\Psi(t-ip)}{X(t-ip)} = \frac{F(t)}{X(t-ip)}, \quad -\infty-i\varepsilon < t < \varepsilon-i\varepsilon \\ &X(z) = \frac{1}{z}X_p(z)X_1(z)X_2(z)\operatorname{sh}\frac{\pi}{2p}zp^{iz/p}\Gamma(1+iz/p) \end{aligned} \tag{14}$$

The functions $X_p(z)$ and $X_2(z)$ are bounded in the whole of the strip and, for sufficiently large $|z|$, the function $X_1(z)$ admits of the estimate

$$|X_1(z)| = O(|t|^{-2\tau/p-1}), \quad z = t+i\tau, \quad -p<\tau<0$$

Hence it follows that

$$X(z) = O(|t|^{-3\tau/p-1/2}), \quad -p<\tau<0$$

Hence, the solution of problem (14) can be represented in the form

$$\Psi(z) = \frac{X(z)}{2ip}\int_{-\infty-i\varepsilon}^{\infty-i\varepsilon}\frac{F(t)dt}{X(t-ip)\operatorname{sh}\frac{\pi}{p}(t-z)}, \quad -p-\varepsilon < \operatorname{Im}z < -\varepsilon \tag{15}$$

Suppose $p \geq 1$. If the function $N_0(t)$ is analytically continuable in the strip $-1 < \mathrm{Im}z < 1$ and vanishes exponentially at infinity, it follows from condition (12) and formula (15) that the function

$$\Psi_1(z) = \begin{cases} \Psi(z), & -p-\varepsilon < \mathrm{Im}z < -\varepsilon \\ \dfrac{F(z)-\Psi(z-ip)}{d_0 z(p+iz)G(z)}, & -\varepsilon < \mathrm{Im}z < p-\varepsilon \end{cases}$$

is holomorphic in the strip $-p-\varepsilon < \mathrm{Im}z < p-\varepsilon$, vanishes exponentially at infinity and is bounded in the whole strip apart from at the points $z_j^+ = t_j^+ + i\tau_j^+$ $(j = 1, 2, \ldots, l)$ which are the zeros of the function $G(z)$ in the strip $-\varepsilon < \mathrm{Im}z < p-\varepsilon$.

Using Cauchy's formula, the required contact stress can be represented in the form

$$\Delta P(x) = P(x) - P_0(x) = \frac{1}{\sqrt{2\pi}x}\int_{-\infty}^{\infty} t\Psi(t)e^{it\ln x}dt = \frac{1}{\sqrt{2\pi}x}\int_{-\infty}^{\infty}(t-ip)\Psi(t-ip)e^{i(t-ip)\ln x}dt$$

Consequently, in the neighbourhood of the vertex of the angle $(x \to 0)$, we have $\Delta P(x) = x^{p-1}g(x)$, where $g(x)$ is a bounded function when $x \geq 0$. For large x, we have $\Delta P(x) = O(x^{-(1+\tau_1^+)})$.

If $0 < p < 1$, the function $\Psi(z)$, which is given by formula (15), is analytically continuable in the strip $-1 < \mathrm{Im}z < -\varepsilon$ except at the points $w_j^- = \lambda_j^- + i\mu_j^-$ $(j = 1, 2, \ldots, q)$ which are the poles of the function $G(z)$ in this strip. In the neighbourhood of the point $x = 0$, the normal contact stress can then be represented in the following manner: $\Delta P(x) = \tilde{c}x^{-(1+\mu_1^-)} + \tilde{g}(x)$, where $\tilde{g}(x)$ is a bounded function when $x \geq 0$, $\tilde{c} = \mathrm{const}$.

We will now consider the case when $p < 2$, that is, the stiffness of the rod increases at the vertex of the angle and decreases at infinity. On introducing the notation $m = -p$ $(m > 0)$, using reasoning similar to that presented above, we can write condition (12) in the form

$$\frac{\Psi_0(t)}{\tilde{X}(t)} + \frac{\Psi_0(t+im)}{\tilde{X}(t+im)} = \frac{F_0(t)}{\tilde{X}(t)}, \quad -\infty + i\varepsilon < t < \infty + i\varepsilon$$

$$\tilde{X}(z) = \frac{1}{z}X_m(z)\kappa(z)(z-im/2)\,\mathrm{sh}\frac{\pi}{2m}z, \quad \varepsilon < \mathrm{Im}z < m+\varepsilon$$

$$X_m(z) = \exp\left\{\frac{1}{2im}\int_{-\infty+i\varepsilon}^{\infty+i\varepsilon}\ln G_m(t)\,\mathrm{cth}\frac{\pi}{m}(t-z)dt\right\} \tag{16}$$

$$\kappa(z) = \exp(-iz/m\ln(ad_0))m^{-3iz/m}\Gamma^2(1+iz/m)/\Gamma(2+iz/m)$$

$$G_m(t) = \frac{t+i}{a(t-im)}\frac{2t-im}{2t+im}\frac{\Delta_1(t)}{\Delta(t)}\mathrm{th}\frac{\pi}{2m}t$$

For sufficiently large $|z|$, the function $\tilde{X}(z)$ admits of the estimate

$$|\tilde{X}(z)| = O(|t|^{3\tau/m-5/2}), \quad 0 < \tau < m$$

The function $\Psi_0(z)/\tilde{X}(z)$ is holomorphic in the strip $\varepsilon < \mathrm{Im}z < m+\varepsilon$ except at the point $z = im/2$ where it can have a first-order pole. The solution of problem (16) is therefore given by the formula

$$\Psi_0(z) = \frac{\tilde{X}(z)}{2im}\int_{-\infty+i\varepsilon}^{\infty+i\varepsilon}\frac{F_0(t)dt}{\tilde{X}(t+im)\,\mathrm{sh}\frac{\pi}{m}(t-z)} + \frac{A_0\tilde{X}(z)}{\mathrm{ch}\frac{\pi}{m}z}$$

$$F_0(z) = d_0(iz-1)(m-iz)(\Delta_1(z)/\Delta(z))N_0(z), \quad A_0 = \mathrm{const}, \quad \varepsilon < \mathrm{Im}z < m+\varepsilon$$

From the equality

$$\int_0^\infty t(P(t) - P_0(t))dt = 0$$

we obtain $\Psi_0(i) = 0$, whence the constant A_0 is also determined.

The function

$$\Psi_2(z) = \begin{cases} \Psi_0(z), & \varepsilon < \mathrm{Im}z < m + \varepsilon \\ \dfrac{F_0(z) + \Psi_0(z + im)}{d_0 z(m - iz)G(z)}, & -m - \varepsilon < \mathrm{Im}z < \varepsilon \end{cases}$$

is holomorphic in the strip $-m + \varepsilon < \mathrm{Im}z < m + \varepsilon$, vanishes at infinity and is continuable in the boundary of the strip except at the points $z_j^- = t_j^- + i\tau_j^-$ $(j = 1, 2, \dots, n)$ which are the zeros of the function $G(z)$ in the strip $-m + \varepsilon < \mathrm{Im}z < \varepsilon$.

If $\tau_1^- < -1$, then the function $\Psi_2(z)$ is analytically continuable in the strip $-1 < \mathrm{Im}z < m + \varepsilon$ and the normal contact stress $\Delta P(x)$ is bounded in the neighbourhood of the point $x = 0$.

If $\tau_1^- > -1$, the function $\Psi_2(z)$ has a pole very close to the real axis at the point $z_1^- = t_1^- + i\tau_1^-$ and, consequently, the contact stress in the neighbourhood of the point $x = 0$ can be represented in the form

$$\Delta P(x) = \tilde{c}_1 x^{-(1+\tau_1^-)} + \tilde{g}_1(x)$$

where $\tilde{g}_1(x)$ is a bounded function when $x \geq 0$, $\tilde{c}_1 = \mathrm{const}$. For large x, we have

$$\Delta P(x) = O(x^{-1-m}), \quad x \to \infty$$

We will now consider some special cases. As will be clear from the subsequent account, in the cases being considered

$$\Delta P(x) = \begin{cases} O(x^{p-1}), & p \geq 1 \\ O(x^{\xi}), & 0 < p < 1; \quad x \to 0 \\ O(x^{\eta}), & p < 0 \end{cases} \tag{17}$$

Suppose the domain S is a half-plane. Then,

$$\theta_1 = \theta_2 = \theta = \pi, \quad \delta = 0, \quad \gamma = 2\pi, \quad \mu = 0$$

$$\Delta(t) = 2|s_1 - s_2|^2 \mathrm{sh}\pi t, \quad \Delta_1(t) = 2|s_1 - s_2|^2 a\,\mathrm{sh}\pi t\,\mathrm{ch}\pi t$$

Hence it follows that $\tau_1^- = -1/2$, $\mu_1^- = -1$ and the function $\Delta P(x)$ satisfies relations (17) when $\xi = 0$, $\eta = -1/2$.

When $\theta = 2\pi$, that is, the plane is cut along the real, positive axis, we obtain

$$\theta_1 = \theta_2 = 2\pi, \quad \delta = 0, \quad \gamma = 4\pi, \quad \mu = 0$$

$$\Delta(t) = 2|s_1 - s_2|^2 \mathrm{sh}2\pi t, \quad \Delta_1(t) = 2|s_1 - s_2|^2 a\,\mathrm{sh}2\pi t\,\mathrm{ch}2\pi t$$

Consequently, $\tau_1^- = -1/4$, $\mu_1^- = -1/2$ and the function $\Delta P(x)$ satisfies relations (17) when $\xi = -1/2$, $\eta = -3/4$.

Note that, when $p = m = 0$, condition (12) gives

$$\Psi(z) = F(z)/(id_0 z^2 G(z) + 1)$$

and the estimate

$$\Delta P(x) = O(x^{\lambda - 1}) \quad \text{when} \quad x \to 0$$

holds for the normal stress, where $\lambda = -\operatorname{Im}\mu$ and μ is the zero of the function $id_0z^2G(z) + 1$ in the lower half-plane which is closest to the real axis.

In the special case, when the body is orthotropic and one of its axes of anisotropy is parallel to the edge of the wedge on which the beam is supported, it has been proved that, when $p < 0$, the normal contact stress in the neighbourhood of the end of the beam is bounded when $\theta \leq \pi/2$ and has the form $\Delta P(x) = O(x^{-\tau_0})$, $x \to 0$ when $\theta > \pi/2$, where $0 < \tau_0 \leq 3/4$. In particular, we have $\Delta P(x) = O(x^{-2/3})$, $x \to 0$ when $\theta = 3\pi/2$.

REFERENCES

1. ALEKSANDROV, V. M. and MKHITARYAN, S. M., *Contact Problems for Bodies with Thin Coatings and Layers*. Nauka, Moscow, 1983.
2. ALEKSANDROV, V. M. and KOVALENKO, Ye. V., *Problems of continuous Mechanics with Mixed Boundary Conditions*. Nauka, Moscow, 1986.
3. POPOV, G. Ya., *Concentration of Elastic Stresses near Punches, Cuts, Fine Inclusions and Reinforcements*. Nauka, Moscow, 1982.
4. BANTSURI, R. D., The contact problem for a wedge with an elastic mounting. *Dokl. Akad. Nauk SSSR*, 1973, **211**, 4, 797–800.
5. POPOV, G. Ya. and TIKHONENKO, L. Ya., The plane problem of the contact of a semi-infinite beam with an elastic wedge. *Prikl. Mat. Mekh.*, 1974, **38**, 2, 312–320.
6. NULLER, B. M., Deformation of an elastic wedge, braced with a girder. *Prikl. Mat. Mekh.*, 1974, **38**, 5, 876–882.
7. BANTSURI, R. D., The contact problem for an anisotropic wedge with an elastic mounting. *Dokl. Akad. Nauk SSSR*, 1975, **222**, 3, 568–571.
8. BANTSURI, R. D., The problem of the bending of a beam lying on an elastic base. *Soobshch. Akad. Nauk Gruz SSR*, 1975, **80**, 2, 317–320.
9. NULLER, B. M., The deformation of an elastic wedge-shaped plate reinforced with a rod of variable stiffness and a method of solving mixed problems. *Prikl. Mat. Mekh.*, 1976, **40**, 2, 306–316.
10. BANTSURI, R. D. and SHAVLAKADZE, N. N., The contact problem for an anisotropic wedge-shaped plate with an elastic mounting of variable stiffness. *Prikl. Mat. Mekh.*, 2002, **66**, 4, 663–669.
11. LEKHNITSKII, S. G., *Anisotropic Plates*. Gostekhizdat, Moscow, 1947.
12. BANTSURI, R. D., A boundary-value problem in the theory of analytic functions, *Soobshch. Akad. Nauk GeorgSSR*, 1974, **73**, 3, 549–552.

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